Cauchy-Goursat theorem

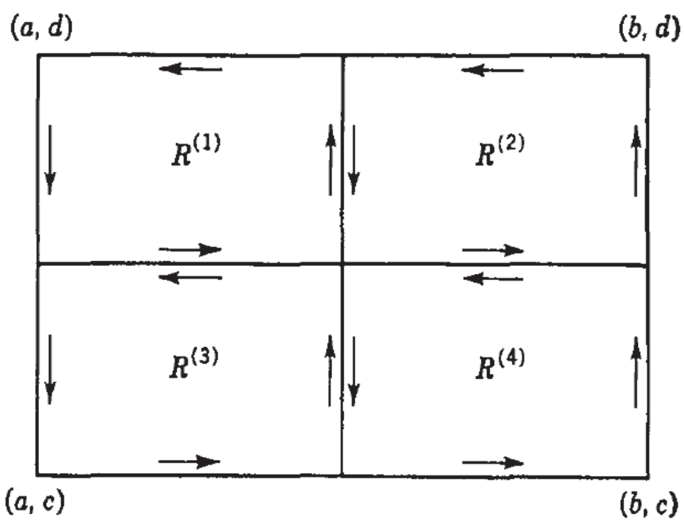
Sunday, September 24, 2023 9:06 AM

Édouard Goursat

Theorem (Cauchy - Goursat).
Let R be a rectangle, $f \in \mathcal{A}(R)$.
Then $\oint_{\partial R} f(z)\,dz = 0$.

Corollary (Cauchy). Let $f \in \mathcal{A}(B(z_0, r))$.
Then for any closed $\gamma \subset B(z_0, r)$,
$\oint_\gamma f(z)\,dz = 0$. There exists $F \in \mathcal{A}(B(z_0, r))$:
$F'(z) = f(z) \quad \forall z \in B(z_0, r)$.

Proof of Theorem.



Key idea: cut R into four equal rectangles: $R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$. Observe:

$$\oint_{\partial R} f(z)\,dz = \sum_{j=1}^{4} \oint_{\partial R^{(j)}} f(z)\,dz$$

$$\ell(\partial R^{(j)}) = \frac{1}{2}\ell(\partial R). \quad \operatorname{diam}(R^{(j)}) = \frac{1}{2}\operatorname{diam} R.$$

Now: assume $\left|\oint_{\partial R} f(z)\,dz\right| = A \neq 0$. Let $R_0 := R$

Then $\exists\, R^{(j)}$: $\left|\oint_{\partial R^{(j)}} f(z)\,dz\right| \geq \frac{A}{4}$. Take $R_1 = R^{(j)}$ for such j.

Repeat: $\exists\, R_2$ - subrectangle of R_1: $\left|\oint_{\partial R_2} f(z)\,dz\right| \geq \frac{1}{4}\left|\oint_{\partial R_1} f(z)\,dz\right| \geq 4^{-2}A$.

R_n - subrectangle of R_{n-1}:

$$\left|\oint_{\partial R_n} f(z)\,dz\right| \geq A\cdot 4^{-n}.$$

$$\ell(\partial R_n) = 2^{-n}\,\ell(R)$$

$$\operatorname{diam}(R_n) = 2^{-n}\operatorname{diam} R.$$

Let $\{z^*\} = \bigcap R_n$ (non-empty, $\operatorname{diam} \bigcap R_n \leq 2^{-n} \operatorname{diam} R\ \forall n \Rightarrow \bigcap R_n$ is one point)

Pick $r>0$: $f \in \mathcal{A}(B(z^*, r))$ (exists, since $z^* \in R$, f is analytic in an open $U \supset R$).

$f'(z_*)$ exists. So $\forall \varepsilon>0\ \exists\, 0<\delta<r$: $|z-z^*|<\delta \Rightarrow \left|\frac{f(z)-f(z^*)}{z-z^*} - f'(z^*)\right| < \varepsilon$.

Or $|f(z) - f(z^*) - f'(z^*)(z-z^*)| < \varepsilon\,|z-z^*|$.

Take n: $2^{-n}\operatorname{diam}(R) < \delta$. Then $\partial R_n \subset B(z^*, \delta)$

$$\oint_{\partial R_n} f(z)\,dz = \overbrace{\oint_{\partial R_n}\left(f(z) - f(z^*) - f'(z^*)(z-z^*)\right)dz}^{\mathrm{I}} +$$

$$\underbrace{\oint_{\partial R_n} f(z^*)\,dz}_{\mathrm{II}} + \underbrace{\oint_{\partial R_n} f'(z^*)\,(z-z^*)\,dz}_{\mathrm{III}}$$

$$f(z^*) = (z f(z^*))'$$

$$f'(z^*)(z-z^*) = \left(\frac{f'(z^*)}{2}(z-z^*)^2\right)', \quad \text{so} \quad \mathrm{II} = \mathrm{III} = 0.$$

$\ldots$, so $II = III = 0$.

$$f'(z^*)(z-z^*) = \left(\frac{f'(z^*)}{2}(z-z^*)^2\right)'$$

But $|f(z) - f(z^*) - f'(z^*)(z-z^*)| < \varepsilon |z - z^*| \le \varepsilon \operatorname{diam}(R_n)$

so

$$4^{-n} A \le \left|\oint_{\partial R_n} f(z)\,dz\right| = |I| \le \left(\varepsilon \operatorname{diam}(R_n)\right)\left(\ell(\partial R_n)\right) = 4^{-n}\varepsilon \operatorname{diam}(R)\,\ell(\partial R)$$

so $4^{-n} A \le \varepsilon\, 4^{-n} \operatorname{diam}(R)\,\ell(\partial R) \quad \forall \varepsilon > 0$

$$A \le \varepsilon \operatorname{diam}(R)\,\ell(\partial R)$$

Assumed $A > 0$. Take $\varepsilon = \dfrac{A}{2\operatorname{diam}(R)\,\ell(\partial R)}$

<u>Get</u>: $A \le \frac{A}{2}$. – contradiction!

<u>Theorem</u> (Improved Goursat)

Let R be a rectangle, $R' := R \setminus \{z_1, \ldots, z_n\}$. Let $f \in A(R')$, and $(z_i \in \operatorname{Int} R)$.

$\lim_{z \to z_i} (z - z_i) f(z) = 0 \quad \forall i$.

Then $\int_{\partial R} f(z)\,dz = 0$.

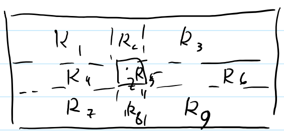
Remark. If f is continuous at z_i, or even locally-bounded ($\exists M, \delta: 0<|z-z_i|<\delta \Rightarrow |f(z)|<M$), then automatically $\lim\limits_{z\to z_i}(z-z_i)f(z)=0$.

Corollary. Let $f\in A(B(z_0,r)\setminus\{z_1,\dots,z_n\})$. Let $\forall i: \lim\limits_{z\to z_i}(z-z_i)f(z)=0$. Then for any closed $\gamma\subset B(z_0,r)\setminus\{z_1,\dots,z_n\}$ $\oint_\gamma f(z)\,dz=0$. There exists $F\in A(B(z_0,r)\setminus\{z_1,\dots,z_n\})$:

$$F'(z)=f(z)\quad \forall z\in B(z_0,r)\setminus\{z_1,\dots,z_n\}$$

Proof of Improved Goursat.

By subdivision, can assume that $n=1$.



R

Cut R into 9 rectangles, as shown.

For $j\neq 5$, $f\in A(R_j)$, so $\oint_{\partial R_j} f(z)\,dz=0$.

Observe: $$\oint_{\partial R} f(z)\,dz=\sum_{j=1}^{9}\oint_{\partial R_j} f(z)\,dz=\oint_{\partial R_5} f(z)\,dz.$$

Fix $\varepsilon>0$. Choose $\delta>0$: $|z-z_1|<\delta \Rightarrow |f(z)(z-z_1)|<\varepsilon \Rightarrow |f(z)|<\dfrac{\varepsilon}{|z-z_1|}$.

Take R_5 to be a square of size $\frac{\delta}{2}$ centered at z_1.

Then $z\in\partial R_5 \Rightarrow \frac{\delta}{4}\le|z-z_1|\le\frac{\delta}{\sqrt{2}}<\delta$, so $z\in\partial R_5 \Rightarrow |f(z)|<\dfrac{\varepsilon}{|z-z_1|}$

So $$\left|\oint_{\partial R} f(z)\,dz\right| = \left|\oint_{\partial R_5} f(z)\,dz\right| \le \varepsilon\int_{\partial R_5}\frac{|dz|}{|z-z_1|}\le \varepsilon\cdot\frac{4}{\delta}\cdot\overset{\ell(\partial R_5)}{\overset{\|}{2\delta}}=8\varepsilon.$$

Since ε is arbitrary, $\oint_{\partial R} f(z)\,dz=0$. ∎

$|z-z_1|\ge\frac{\delta}{4}$